Universal Solutions of Quantum Dynamical Yang–Baxter Equations

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Abstract

We construct a universal trigonometric solution of the Gervais–Neveu–Felder equation in the case of finite dimensional simple Lie algebras and finite dimensional contragredient simple Lie superalgebras.

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1 Introduction

Let $\mathfrak{H}$ be a finite dimensional commutative Lie algebra over $\mathbb{C}$, V a semisimple finite dimensional $\mathfrak{H}-$module. The quantum dynamical Yang–Baxter equation also known as Gervais–Neveu–Felder equation (GNF) is:

$$R_{12}(-\mu + 2h^{(3)})R_{13}(-\mu)R_{23}(-\mu + 2h^{(1)}) = R_{23}(-\mu)R_{13}(-\mu + 2h^{(2)})R_{12}(-\mu) \qquad (1)$$

where $R : \mathfrak{H}^* \to \mathrm{End}(V \otimes V)$ is a function (often chosen to be meromorphic) and where by definition

$$R_{12}(-\mu + 2h^{(3)})(v_1 \otimes v_2 \otimes v_3) = R_{12}(-\mu + 2\eta)(v_1 \otimes v_2) \otimes v_3 \qquad (2)$$

if v_3 has weight η.

This equation was first discovered by J.L. Gervais and A. Neveu in their work on the quantisation of Liouville theory [12]. It reappeared more recently in its modern form in the work of Felder [10] in his approach to the quantisation of Knizhnik–Zamolodchikov–Bernard equations. Since then, it has been shown to be one of the basic tools in the R-matrix formalism of the quantisation of a wide family of models (Calogero–Moser, Calogero–Sutherland, Ruijsenaars–Schneider) [1].

The aim of our work is to associate to each finite dimensional simple Lie algebra (and also to each finite dimensional contragredient simple Lie superalgebra) a trigonometric universal solution of GNF.

Let $\mathfrak{G}$ be a simple Lie algebra over $\mathbb{C}$ of rank r and $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{G}$. Denote by (ℓ_i) an orthonormal basis of $\mathfrak{H}$ with respect to the Killing form and $\ell^i \in \mathfrak{H}^\star$ its dual basis. A universal solution of GNF is a meromorphic map $R : \mathbb{C}^r \to \mathfrak{U}_q(\mathfrak{G})^{\otimes 2}$ solving the equation

$$R_{12}(xq^{2\ell^{(3)}})R_{13}(x)R_{23}(xq^{2\ell^{(1)}}) = R_{23}(x)R_{13}(xq^{2\ell^{(2)}})R_{12}(x) \qquad (3)$$

where $x = (x_1, \cdots, x_r) \in \mathbb{C}^r$, and $xq^\ell = (x_1 q^{\ell_1}, \cdots, x_r q^{\ell_r})$. Note that if R is meromorphic there is no difficulty in defining $R_{12}(xq^{2\ell^{(3)}})$.

If π is a finite dimensional representation of $\mathfrak{U}_q(\mathfrak{G})$ acting on V then $(\pi \otimes \pi)(R(x))$ is a solution of GNF with the change of variable $x_i = q^{-\mu_i}$ where $\mu = \sum_i \mu_i \ell^i$.

A universal solution of GNF equation can be obtained from a solution of the shifted cocycle equation (4). Let us assume that $F : \mathbb{C}^r \to \mathfrak{U}_q(\mathfrak{G})^{\otimes 2}$ is an invertible element of weight 0, i.e. $[F_{12}, h \otimes 1 + 1 \otimes h] = 0, \forall h \in \mathfrak{H}$, satisfying the following equation:

$$(\Delta \otimes \mathrm{id})(F(x)) \; F_{12}(xq^{\ell^{(3)}}) = (\mathrm{id} \otimes \Delta)(F(x)) \; F_{23}(x) \qquad (4)$$

Then $R(x) = F_{21}(x)^{-1}R_{12}F_{12}(x)$ is easily shown to satisfy the universal GNF equation where R is a universal element in $\mathfrak{U}_q(\mathfrak{G})^{\otimes 2}$ satisfying the quasi-triangularity axioms.

An explicit formula for R is known for every $\mathfrak{U}_q(\mathfrak{G})$ [8, 17, 7, 15, 16], see also [14] in the case of quantum superalgebras. As far as $R(x)$ is concerned, matricial solutions of GNF have been given in [5] for $sl(n)$. Elliptic matricial solutions of GNF with spectral parameter have been exhibited in [10] and partially classified in [9]. Explicit solutions to the shifted cocycle are known in the $sl(2)$ case [2] and also in the $osp(1|2)$ case [4]. Unicity of the solution of the shifted cocycle equation under some hypothesis and recursion relations have been written in [11] for general simple Lie algebra. An important formula $F(x) = \prod_{k=0}^{+\infty} F_k$ has been obtained, and F_k has been exactly computed in the case of $sl(2)$. In the rest of this work we will give exact formulas for F_k and provide an alternative approach to the computation of $F(x)$ (using a linear equation) and to the proof of the cocycle identity (performing an algebraic as well as an analytical study).

2 Notations

In the following we will always assume that $q \in \mathbb{R}$, with $0 < q < 1$. We will denote by κ the restriction of the Killing form to $\mathfrak{H}$. If $\alpha \in \mathfrak{H}^\star$ we will denote by $t_\alpha \in \mathfrak{H}$ the element defined by $\kappa(t_\alpha, h) = \alpha(h), \forall h \in \mathfrak{H}$, and by $(.|.)$ the scalar product on $\mathfrak{H}^\star$ defined from κ by duality. Let $(\alpha_i, i = 1, \ldots, r)$ be a choice of simple roots, Φ the set of roots and Φ^+ the corresponding set of positive roots. To each root α we will associate the element $h_\alpha = \frac{2}{(\alpha|\alpha)} t_\alpha$.

A presentation of $\mathfrak{U}_q(\mathfrak{G})$ by generators and relations is given by:

$$[t_{\alpha_i}, t_{\alpha_j}] \; = \; 0 \qquad\qquad [e_{\alpha_i}, f_{\alpha_j}] \; = \; \delta_{ij} \, \frac{q^{t_{\alpha_i}} - q^{-t_{\alpha_i}}}{q - q^{-1}} \tag{5}$$

$$\begin{aligned}{} [t_{\alpha_i}, e_{\alpha_j}] &= a_{ij}^{\mathrm{sym}} e_{\alpha_j} \\ [t_{\alpha_i}, f_{\alpha_j}] &= -a_{ij}^{\mathrm{sym}} f_{\alpha_j} \end{aligned} \qquad \text{with} \quad a_{ij}^{\mathrm{sym}} = (\alpha_i | \alpha_j) \tag{6}$$

$$\begin{aligned}{} (\mathrm{ad}_{q'} e_{\alpha_i})^{n_{ij}} (e_{\alpha_j}) &= 0 \\ (\mathrm{ad}_{q'} f_{\alpha_i})^{n_{ij}} (f_{\alpha_j}) &= 0 \end{aligned} \qquad \text{if} \quad i \neq j \text{ and } n_{ij} = 1 - 2\frac{a_{ij}^{\mathrm{sym}}}{a_{ii}^{\mathrm{sym}}}, \; q' = q \text{ or } q^{-1} \tag{7}$$

where we have introduced:

$$(\mathrm{ad}_{q^{\pm 1}} x)(y) = \sum_{(x)} x_{(1)} \, y \, S^{\pm 1}(x_{(2)}). \tag{8}$$

We will call $\mathfrak{U}_q(\mathfrak{N}_+)$ (resp. $\mathfrak{U}_q(\mathfrak{N}_-)$) the subalgebra of $\mathfrak{U}_q(\mathfrak{G})$ generated by e_{α_i}, $i = 1, \ldots, r$ (resp. f_{α_i}, $i = 1, \ldots, r$). As usual, we denote by $\mathfrak{U}_q(\mathfrak{B}_+)$ (resp. $\mathfrak{U}_q(\mathfrak{B}_-)$) the algebra generated by h_{α_i}, e_{α_i} $i = 1, \ldots, r$ (resp. h_{α_i}, f_{α_i} $i = 1, \ldots, r$).

The Hopf algebra structure is defined by:

$$\begin{aligned} \Delta(t_{\alpha_i}) &= t_{\alpha_i} \otimes 1 + 1 \otimes t_{\alpha_i} \tag{9} \\ \Delta(e_{\alpha_i}) &= e_{\alpha_i} \otimes q^{t_{\alpha_i}} + 1 \otimes e_{\alpha_i} \tag{10} \\ \Delta(f_{\alpha_i}) &= f_{\alpha_i} \otimes 1 + q^{-t_{\alpha_i}} \otimes f_{\alpha_i} \tag{11} \end{aligned}$$

Let us fix a normal ordering $<$ on the set of positive roots (see [14] and references therein). Such a normal ordering is defined by requiring that for α, β, $\alpha + \beta$ positive roots such that $\alpha < \beta$, we have $\alpha < \alpha + \beta < \beta$. For a given set of positive roots, there exists several normal orderings. It can be shown that they are related by elementary moves recalled in [14]. Then, to each non-simple positive root α we can associate elements (which depend on the choice of the normal ordering) $e_\alpha \in \mathfrak{U}_q(\mathfrak{N}_+)$ and $f_\alpha \in \mathfrak{U}_q(\mathfrak{N}_-)$. They are uniquely defined by:

$$e_{\alpha+\beta} \; = \; e_\alpha e_\beta - q^{-(\alpha|\beta)} e_\beta e_\alpha \text{ and } f_{\alpha+\beta} \; = \; f_\beta f_\alpha - q^{(\alpha|\beta)} f_\alpha f_\beta \tag{12}$$

for $\alpha < \beta$ such that there is no pair $\{\alpha', \beta'\}$ in $]\alpha, \beta[$ with $\alpha + \beta = \alpha' + \beta'$. It can be shown that

$$[e_\alpha, f_\alpha] = a_\alpha \, \frac{q^{t_\alpha} - q^{-t_\alpha}}{q - q^{-1}} \tag{13}$$

where a_α are functions of q.

A Poincaré–Birkhoff–Witt basis of $\mathfrak{U}_q(\mathfrak{N}_+)$ (resp. $\mathfrak{U}_q(\mathfrak{N}_-)$) is given by: $(e^p = \prod_{\alpha \in \Phi^+}^{>} (e_\alpha)^{p_\alpha})_{p \in \mathcal{Z}}$ (resp. $(f^r = \prod_{\alpha \in \Phi^+}^{>} (f_\alpha)^{r_\alpha})_{r \in \mathcal{Z}}$), where $\mathcal{Z} = \mathrm{Map}(\Phi^+, \mathbb{Z}^+)$ and the products are ordered according to $>$, the reversed normal order on Φ_+.

As a notation, for $p \in \mathcal{Z}$, we introduce $\rho_p = \frac{1}{2} \sum_{\alpha \in \Phi^+} p_\alpha \alpha$ which satisfies

$$[h, e^p] = \rho_p(h) \, e^p \text{ and } [h, f^p] = -\rho_p(h) \, f^p, \quad \forall h \in \mathfrak{H}. \tag{14}$$

As usual, we denote by $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ the Weyl vector. $\mathfrak{U}_q(\mathfrak{G})$ is $\mathbb{Z}$-graded with respect to the adjoint action of t_ρ, which is called the principal gradation of $\mathfrak{U}_q(\mathfrak{G})$. The principal degree of the generators e_{α_i} is 1.

$\mathfrak{U}_q(\mathfrak{G})$ is quasi-triangular and let $R \in (\mathfrak{U}_q(\mathfrak{B}_+) \otimes \mathfrak{U}_q(\mathfrak{B}_-))^c$ be a solution of the quasi-triangularity conditions. We have denoted with a superscript c a completion of $\mathfrak{U}_q(\mathfrak{B}_+) \otimes \mathfrak{U}_q(\mathfrak{B}_-)$ in the usual sense: elements of $(\mathfrak{U}_q(\mathfrak{B}_+) \otimes \mathfrak{U}_q(\mathfrak{B}_-))^c$ are of the form $g = \sum_{p,r \in \mathcal{Z}} (e_p \otimes f_r) \psi_{p,r}(x)$ where $\psi_{p,r}(x)$ are arbitrary functions of $h_{(1)}$ and $h_{(2)}$, i.e. are elements of $\mathrm{Map}(\mathfrak{H}^* \oplus \mathfrak{H}^*, \mathbb{C})$. It can be shown that we have the explicit multiplicative formula

$$R = K\hat{R} \quad \text{where} \quad K = \prod_{j=1}^{r} q^{\ell_j \otimes \ell_j} \quad \text{and} \quad \hat{R} = \prod_{\alpha \in \Phi^+}^{>} \hat{R}_\alpha \tag{15}$$

and $\hat{R}_\alpha = \exp_{\bar{q}_\alpha}((q - q^{-1}) a_\alpha^{-1} e_\alpha \otimes f_\alpha)$ with $\bar{q}_\alpha = q_\alpha^{-(\alpha|\alpha)}$. The q-exponential is defined by:

$$\exp_q(z) = \sum_{n=0}^{+\infty} \frac{1}{[n]_q!} z^n \quad \text{with} \quad [n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q \quad \text{and} \quad [n]_q = \frac{1 - q^n}{1 - q}. \tag{16}$$

An important result states that the value of R is independent of the choice of normal ordering.

Remark: We have essentially used the conventions of [14], except that we have applied to all the expressions therein the antimorphism and comorphism † defined by

$$\underline{e}_\alpha^\dagger = f_\alpha \ , \ \underline{f}_\alpha^\dagger = e_\alpha \text{ and } \underline{h}_\alpha^\dagger = t_\alpha \ \forall \alpha \in \Phi_+ \tag{17}$$

where the underlined expressions denote the bases used in [14]. The R matrix we use is $R_{12} = (\dagger \otimes \dagger)(\underline{R}_{21})$.

3   Universal solution in the case of Lie algebras

The aim of our work is to show that, hidden in the resolution of the shifted cocycle equation, there is a linear equation satisfied by F, the solution of which is unique under some hypothesis. This linear equation was shown to be satisfied by $F(x)$ in the $sl(2)$ case in [6]. The necessity of this equation came from a complete different point of view: in [6] it has been shown that the characters of the irreducible unitary representations of the quantum Lorentz group are constructed from complex continuation of $6j$ symbols of $\mathfrak{U}_q(su(2))$. It has been shown in [3] that there is a complete dictionary between matrix elements of $F(x)$ and complex continuation of $6j$ symbols. From the regularity conditions of the characters of the quantum Lorentz group and the explicit expression of F, a linear equation satisfied by F was derived.

Let $B(x) \in \mathfrak{U}_q(\mathfrak{H})^c = \mathrm{Map}(\mathfrak{H}^*, \mathbb{C})$ be the element $B(x) = \prod_{j=1}^{r} (x_j^{\ell_j} q^{\ell_j^2}) = q^{\sum_{j=1}^{r} (\ell_j \ell_j - \mu_j \ell_j)}$, the generalisation of the linear equation of [6] can be written:

$$F_{12}(x) B_2(x) = \hat{R}_{12}^{-1} B_2(x) F_{12}(x) \tag{18}$$

We first have the following proposition on the existence and unicity of the solution of this linear equation.

Proposition 1:

Under the following hypothesis:

    1. $F(x) \in \left(\mathfrak{U}_q(\mathfrak{B}^+) \otimes \mathfrak{U}_q(\mathfrak{B}^-)\right)^c$

    2. the projection of $F(x)$ on $\left(\mathfrak{U}_q(\mathfrak{H})^{\otimes 2}\right)^c$ is equal to $1 \otimes 1$

there exists a unique solution of equation (18).
Moreover, under these assumptions, we have

$$F(x) = \sum_{p,r \in \mathcal{Z}} e^p \otimes \left(f^r \ \phi_{p,r}(x)\right) \tag{19}$$

where $\phi_{p,r}(x)$ are rational functions on $\mathfrak{H}$.
Under these conditions F is of weight zero i.e. $[F(x), h \otimes 1 + 1 \otimes h] = 0, \ \forall h \in \mathfrak{H}$.

<u>Proof:</u> Starting with $\hat{R}_{12}^{-1} = \sum_{s \in \mathcal{Z}} \sigma_s \ e^s \otimes f^s$ and $F(x) = \sum_{p,r \in \mathcal{Z}} \left(e^p \otimes f^r\right) \psi_{p,r}(x)$, where $\psi_{p,r}(x)$ a priori belongs to $\mathrm{Map}(\mathfrak{H}^* \oplus \mathfrak{H}^*, \mathbb{C})$, one computes that the equation (18) can be rewritten as

$$\left(1 - q^{(\mu + 2\rho_r | 2\rho_r)} q^{-4t_{\rho_r}^{(2)}}\right) \psi_{p,r}(x) = \sum_{\substack{k+s=p \\ l+s=r \\ s \neq 0}} a_p^{ks} b_r^{ls} \sigma_s q^{(\mu + 2\rho_l | 2\rho_l)} q^{-4t_{\rho_l}^{(2)}} \psi_{k,l}(x) \tag{20}$$

where we have introduced the numbers a_p^{ks} and b_r^{ls} defined by

$$e^k e^s \ = \ \sum_{p \in \mathcal{Z}} a_p^{ks} e^p \ \text{ and } \ f^k f^s \ = \ \sum_{p \in \mathcal{Z}} b_p^{ks} f^p$$

The equation (20) clearly shows that we have a linear system which is strictly triangular with respect to the principal gradation. Moreover, once we have normalised $\psi_{0,0}(x) = 1$, the equation proves that $\psi_{p,r}$ are rational functions on $1 \otimes \mathfrak{H}$ which are uniquely defined.
Now, if $F_{12}(x)$ is a solution of (18) satisfying the two hypothesis of proposition 1, for any $\lambda \in \mathbb{C}$ and any element $h \in \mathfrak{H}$, $\tilde{F}_{12}(x) = \lambda^{h_{(1)} + h_{(2)}} F_{12}(x) \lambda^{-(h_{(1)} + h_{(2)})}$ is also a solution of (18) satisfying the two hypothesis of proposition 1. Thus, as the solution is unique, we must have $\tilde{F}_{12}(x) = F_{12}(x)$, so that F is of zero weight. $\qquad \square$

Although we have not mentioned up to now the problems related to the convergence of the series (19), it suffices to say that in each finite dimensional representation this series has a finite number of terms. In the sequel when we will write a series or an infinite product we will always assume that they are evaluated in finite dimensional representation, and we will therefore only have to consider the convergence of a series or infinite product of finite dimensional matrices.

We now show that the unique solution of (18) satisfying the relations (4) can be written as an infinite and very explicit product. Let us define the formal product

$$F = \prod_{k=0}^{+\infty} B_2^k \hat{R}_{12}^{-1} B_2^{-k} \tag{21}$$

the convergence of which is analysed in the proposition 5. We will show that $F(x)$ satisfies the linear equation (18) and moreover satisfies the cocycle identity (4). This proposition is divided in two parts: an algebraic study and an analytic one where the limit $N \to +\infty$ is analysed.

Let us define, for each $N \in \mathbb{N}$, $\overset{N}{F}(x) = \prod_{k=0}^{N} B_2^k(x) \hat{R}_{12}^{-1} B_2^{-k}(x)$.

Proposition 2:

$\overset{N}{F}(x)$ *is the unique sequence of elements of* $\mathfrak{U}_q(\mathfrak{G})^{\otimes 2}$ *satisfying the recursion equation*

$$\overset{N}{F}(x)B_2(x) = \hat{R}_{12}^{-1} B_2(x) \overset{N-1}{F}(x) \tag{22}$$

with $\overset{0}{F}(x) = \hat{R}_{12}^{-1}$.

Proof: Trivial computation. □

Remark: we will show in the proposition 5 that, evaluated in the tensor product of any finite dimensional representation, the product (21) is convergent for μ dominant weight with all the scalar products $(\mu|\alpha_i)$ sufficiently large. As a result, in this domain, one can show from (22) that $\lim_{N \to +\infty} \overset{N}{F}(x)$ satisfies the linear equation, and therefore is equal to the expression (19).

Proposition 3:

Let us define $U = B_1 B_2 K_{12}^2$ *and* $X = B_2 K_{12}^2$. *We have the following two identities:*

$$\overset{N}{F}_{23}(x) \overset{N}{F}_{12}^{-1}(xq^{\ell^{(3)}}) = \left(\prod_{k=0}^{N} B_3^k R_{23}^{-1} K_{23} B_3^{-k} \right) \left(\prod_{k=N}^{0} U_{23}^k K_{12}^{-1} R_{12} U_{23}^{-k} \right) \tag{23}$$

$$(\mathrm{id} \otimes \Delta)(\overset{N}{F^{-1}}(x)) \; (\Delta \otimes \mathrm{id})(\overset{N}{F}(x)) = \tag{24}$$
$$\left(\prod_{k=0}^{N} B_3^k R_{23}^{-1} (X_{23}^{N+1} \hat{R}_{13}^{-1} X_{23}^{-N-1}) K_{23} B_3^{-k} \right) \left(\prod_{k=N}^{0} U_{23}^k K_{12}^{-1} (B_3^{N+1} \hat{R}_{13} B_3^{-N-1}) R_{12} U_{23}^{-k} \right)$$

Proof: The first relation is an immediate application of the relation $B_2(xq^{\ell^{(3)}}) = B_2(x) K_{23}^2$. The second relation is more tricky. Let us define $V = K_{12}^{-1} K_{13}^{-1} R_{13} R_{12}$ and $W = R_{23}^{-1} R_{13}^{-1} K_{13} K_{23} B_3$, we have the following identity:

$$U_{23}^{-1} VW = W U_{23}^{-1} V. \tag{25}$$

This identity is shown to be equivalent to Yang–Baxter equation with the use of the two relations:

$$R_{12} K_{13} K_{23} = K_{13} K_{23} R_{12} \text{ and } R_{12} U_{12} = U_{12} R_{12} \tag{26}$$

From the definition of $\overset{N}{F}$ we have

$$(\mathrm{id} \otimes \Delta)(\overset{N}{F^{-1}}) = \prod_{k=N}^{0} U_{23}^k V U_{23}^{-k} = U_{23}^{N+1} (U_{23}^{-1} V)^{N+1} \tag{27}$$

$$(\Delta \otimes \mathrm{id})(\overset{N}{F}) = \prod_{k=0}^{N} B_3^k W B_3^{-k-1} = W^{N+1} B_3^{-N-1}. \tag{28}$$

We can therefore write, using the identity (25):

$$(\mathrm{id} \otimes \Delta)(\overset{N}{F}{}^{-1}) \, (\Delta \otimes \mathrm{id})(\overset{N}{F}) \;=\; U_{23}^{N+1} W^{N+1} (U_{23}^{-1}V)^{N+1} B_3^{-N-1} \tag{29}$$

$$= (U_{23}^{N+1} W U_{23}^{-N-1})^{N+1} \, U_{23}^{N+1} \, (U_{23}^{-1}V)^{N+1} \, B_3^{-N-1} \tag{30}$$

$$= \left(R_{23}^{-1} X_{23}^{N+1} \hat{R}_{13}{}^{-1} X_{23}^{-N-1} K_{23} B_3 \right)^{N+1} U_{23}^{N+1} (U_{23}^{-1}V)^{N+1} B_3^{-N-1} \tag{31}$$

$$= \left(\prod_{k=0}^{N} B_3^k R_{23}^{-1} (X_{23}^{N+1} \hat{R}_{13}^{-1} X_{23}^{-N-1}) K_{23} B_3^{-k} \right) B_3^{N+1} U_{23}^{N+1} (U_{23}^{-1}V)^{N+1} B_3^{-N-1} \tag{32}$$

$$= \left(\prod_{k=0}^{N} B_3^k R_{23}^{-1} (X_{23}^{N+1} \hat{R}_{13}^{-1} X_{23}^{-N-1}) K_{23} B_3^{-k} \right) \left(\prod_{k=N}^{0} U_{23}^k B_3^{N+1} V B_3^{-N-1} U_{23}^{-k} \right) \tag{33}$$

which ends the proof of this proposition. $\qquad\square$

From this result, in order to prove that $\lim_{N\to+\infty} \overset{N}{F}(x)$ satisfies the cocycle identity, it is sufficient to show that both $B_3^{N+1} \hat{R}_{13} B_3^{-N-1}$ and $X_{23}^{N+1} \hat{R}_{13}{}^{-1} X_{23}^{-N-1}$ tend to 1 sufficiently fast. This is indeed the case for the solution (21) in each finite dimensional representation of $\mathfrak{U}_q(\mathfrak{G})$.

We will need the following proposition:

Proposition 4:

1. *Let $\mathfrak{A}$ be a Banach algebra with norm $\|.\|$, and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of elements of $\mathfrak{A}$. A sufficient condition for the product $\prod_{k=0}^{+\infty} u_k$ to be convergent is:*

   - $\exists C > 0, \forall n > 0, \forall m \leq n, \; \sum_{k=m}^{n} \log(\|u_k\|) \leq C$.

   - $\sum_{k=0}^{+\infty} \|u_k - 1\|$ *is convergent.*

2. *Let $(v_k^{(n)})_{(k,n)\in\mathbb{N}^2}$ be a sequence of elements of $\mathfrak{A}$. The product $\prod_{k=0}^{n} u_k v_k^{(n)}$ converges to $\prod_{k=0}^{+\infty} u_k$ if the previous assumptions are satisfied, together with*

   - $\exists C > 0, \forall n > 0, \forall m \leq n, \; \prod_{k=m}^{n} \|v_k^{(n)}\| \leq C$

   - $\exists C' > 0, \forall n > 0, \forall k \leq n, \; \|v_k^{(n)} - 1\| \leq \frac{C'}{n^2}$.

   These sufficient assumptions are of course not at all minimal but we will only need these crude hypothesis.

<u>Proof:</u> The first part of this proposition is a direct application of Cauchy criterion to the partial product $\prod_{k=0}^{n} u_k$. The second one is a direct application of the inequality:

$$\| \prod_{k=0}^{n} u_k v_k^{(n)} - \prod_{k=0}^{n} u_k \| \leq \prod_{l=0}^{n} \|u_l\| \sum_{k=0}^{n} \|v_k^{(n)} - 1\| \, \|v_{k+1}^{(n)}\| \cdots \|v_n^{(n)}\| \tag{34}$$

$\qquad\square$

We will now apply this proposition to the case of the sequence $u_k = B_2^k(x) \hat{R}_{12}^{-1} B_2^{-k}(x)$

Proposition 5:

The product $\prod_{k=0}^{+\infty} B_2^k(x) \hat{R}_{12}^{-1} B_2^{-k}(x)$ in each finite dimensional representation of $\mathfrak{U}_q(\mathfrak{G})$ is convergent for μ such that all the scalar products $(\mu|\alpha_i)$ are sufficiently large.

<u>Proof:</u> Let π_1 and π_2 be finite dimensional representations of $\mathfrak{U}_q(\mathfrak{G})$. By iterative application of the proposition 4 to $u_k = \prod_{\alpha \in \Phi^+}^> B_2^k \hat{R}_\alpha^{-1} B_2^{-k}$, it is sufficient to apply the first criterion of proposition 4 to the sequences $u_k^\alpha = (\pi_1 \otimes \pi_2) \left(B_2^k \hat{R}_\alpha^{-1} B_2^{-k} \right)$.

Then, it is easy to compute that

$$u_k^\alpha = \exp_{\bar{q}_\alpha} \left(-a_\alpha^{-1}(q - q^{-1}) q^{-k(\alpha|\alpha - \mu)} (\pi_1 \otimes \pi_2) \left(e_\alpha \otimes q^{-2kt_\alpha} f_\alpha \right) \right)$$

For $q \in]0, 1[$, $\|u_k^\alpha\| \le \exp_{\bar{q}_\alpha} \left(\epsilon_\alpha q^{k\{(\mu - \alpha|\alpha) - 2\|\pi_2(t_\alpha)\|\}} \right)$, with $\epsilon_\alpha = \left| a_\alpha^{-1}(q - q^{-1}) \right| \|(\pi_1 \otimes \pi_2)(e_\alpha \otimes f_\alpha)\|$. Therefore,

$$\log \left(\prod_{k=m}^{k=n} \|u_k^\alpha\| \right) \le \sum_{k=m}^{k=n} w_k^\alpha \text{ with } w_k^\alpha = \log \left(\exp_{\bar{q}_\alpha} \left(\epsilon_\alpha q^{k\{(\mu - \alpha|\alpha) - 2\|\pi_2(t_\alpha)\|\}} \right) \right) \tag{35}$$

Now, if $(\mu|\alpha) > (\alpha|\alpha) + 2\|\pi_2(t_\alpha)\|$, we have $w_k^\alpha \underset{k \to +\infty}{\sim} \epsilon_\alpha q^{k\{(\mu - \alpha|\alpha) - 2\|\pi_2(t_\alpha)\|\}}$. Thus, $\sum_k w_k^\alpha$ is convergent, which proves from the inequality (35), that the partial sums $\sum_{k=m}^n \log(\|u_k^\alpha\|)$ are bounded.

Moreover, $\|u_k^\alpha - 1\| \le \left(\exp_{\bar{q}_\alpha} \left(\epsilon_\alpha q^{k\{(\mu - \alpha|\alpha) - 2\|\pi_2(t_\alpha)\|\}} \right) - 1 \right) \underset{k \to +\infty}{\sim} \epsilon_\alpha q^{k\{(\mu - \alpha|\alpha) - 2\|\pi_2(t_\alpha)\|\}}$. As a result, $\sum_{k=0}^{+\infty} \|u_k^\alpha - 1\|$ is convergent. $\qquad\square$

Proposition 6:

> F as defined by (21) satisfies the shifted cocycle equation (4) in each finite dimensional representation of $\mathfrak{U}_q(\mathfrak{G})$.

<u>Proof:</u> The two products appearing in the right-hand-side of equation (24) can be rewritten as

$$\prod_{k=0}^N B_3^k \hat{R}_{23}^{-1} X_{23}^{N+1} \hat{R}_{13}^{-1} X_{23}^{-N-1} K_{23} B_3^{-k} = \prod_{k=0}^N B_3^k R_{23}^{-1} K_{23} B_3^{-k} \, v_{k,N} \tag{36}$$

$$\prod_{k=N}^0 U_{23}^k K_{12}^{-1} B_3^{N+1} \hat{R}_{13} B_3^{-N-1} R_{12} U_{23}^{-k} = \prod_{k=N}^0 U_{23}^k K_{12}^{-1} R_{12} U_{23}^{-k} \, w_{k,N} \tag{37}$$

where $v_{k,N} = \prod_{\alpha \in \Phi_+}^> v_{k,N}^\alpha$ and $w_{k,N} = \prod_{\alpha \in \Phi_+}^> w_{k,N}^\alpha$ with

$$v_{k,N}^\alpha = K_{23}^{-1} B_3^k (X_{23}^{N+1} (\hat{R}_\alpha)_{13}^{-1} X_{23}^{-N-1}) K_{23} B_3^{-k}$$
$$w_{k,N}^\alpha = U_{23}^k K_{12}^{-1} B_3^{N+1} K_{13}^{-1} (\hat{R}_\alpha)_{13} B_3^{-N-1} K_{12} U_{23}^{-k}.$$

In order to prove the above proposition, it is sufficient to show that the two sequences $v_{k,N}$ and $w_{k,N}$ obey to the criterions of proposition 4.2 for μ such that all the scalar products $(\mu|\alpha_i)$ are sufficiently large.

As in proposition 5, it is sufficient to show that the sequences $v_{k,N}^\alpha$ and $w_{k,N}^\alpha$ satisfy this criterion. An easy computation leads to

$$v_{k,N}^\alpha = \exp_{\bar{q}_\alpha} \left(-a_\alpha^{-1}(q - q^{-1}) q^{-(N+1+k)(\alpha|\alpha - \mu)} q^{-2(N+1+k)t_\alpha^{(3)} - (2N+1)t_\alpha^{(2)}} e_\alpha \otimes 1 \otimes f_\alpha \right)$$

$$w_{k,N}^\alpha = \exp_{\bar{q}_\alpha} \left(a_\alpha^{-1}(q - q^{-1}) q^{-(N+1+k)(\alpha|\alpha - \mu)} q^{-2(N+1+k)t_\alpha^{(3)} - (2k+1)t_\alpha^{(2)}} e_\alpha \otimes 1 \otimes f_\alpha \right)$$

For each finite dimensional representation π, let us denote $K_\pi = \|q^{-\pi(t_\alpha)}\|$. If π_1, π_2 and π_3 are finite dimensional representations of $\mathfrak{U}_q(\mathfrak{G})$

$$\|\pi_{123}(v_{k,N}^\alpha) - 1\| \leq C_{123} \; q^{-(N+k+1)(\alpha|\alpha-\mu)} \; (K_{\pi_1})^{2(N+k+1)} \; (K_{\pi_3})^{2k+1} \text{ with } \pi_{123} = \pi_1 \otimes \pi_2 \otimes \pi_3$$

Thus, for μ such that all the scalar products $(\mu|\alpha_i)$ are sufficiently large, there exists $r \in]0,1[$ and $K_0 > 0$ such that $\|\pi_{123}(v_{k,N}^\alpha) - 1\| \leq K_0 \; r^N$, which proves that $v_{k,N}^\alpha$ obeys to the second hypothesis of proposition 4.2. Now, from $\|\pi_{123}(v_{k,N}^\alpha)\| \leq 1 + \|\pi_{123}(v_{k,N}^\alpha) - 1\| \leq 1 + K_0 \; r^N$, one easily shows that $v_{k,N}^\alpha$ also satisfies to the first condition. This ends the demonstration for the sequence $v_{k,N}^\alpha$. For $w_{k,N}^\alpha$, the proof follows the same steps. $\qquad\square$

> Proposition 7:
>
> A universal solution of GNF is given by
>
> $$R(x) \;=\; \left(\prod_{k=+\infty}^{0} B_1^k \hat{R}_{21} B_1^{-k} \right) R_{12} \left(\prod_{k=0}^{+\infty} B_2^k \hat{R}_{12}^{-1} B_2^{-k} \right).$$
>
> This solution satisfies
>
> $$R_{12}(x) B_2(x) R_{21}(x) = B_2(x) K_{12}^2.$$

Proof: Trivial computation using the linear equation satisfied by F.

$\qquad\square$

Example: case of $sl(2)$

In [2], a solution to the shifted cocycle equation was constructed. It reads

$$F_{12}(x) \;=\; \sum_{n=0}^{+\infty} \frac{(q - q^{-1})^n}{[n]_{q^2}!} \; e^n \otimes f^n \frac{(-1)^n}{\prod_{\nu=1}^{n} \left(1 - x^{-2} q^{2\nu} q^{-2h_{(2)}}\right)}. \tag{38}$$

This solution satisfies the linear equation [6] and therefore can also be written as, for x sufficiently large,

$$F(x) = \prod_{k=0}^{+\infty} B_2(x)^k \hat{R}_{12}^{-1} B_2(x)^{-k} = \prod_{k=0}^{+\infty} \exp_{q^2}\left(-(q - q^{-1}) \; x^{-2k} q^{-2k} q^{-2kh_{(2)}} \; e \otimes f\right). \tag{39}$$

This last formula appeared in [11][1].

4 Universal solution for contragredient Lie superalgebras

The above construction can be applied to the case of finite dimensional Lie superalgebras with the following modifications. We will only consider contragredient Lie superalgebras, i.e. finite dimensional classical (simple) Lie superalgebras which admits a unique non degenerate invariant

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[1]However, contrary to what has been stated in this work, it cannot be written as $\exp_{q^{-2}}\left(\phi(x, h_{(2)}) \; e \otimes f\right)$.

supersymmetric bilinear form $(.|.)$ [13]. These are formed by the $A(m,n)$, $B(m,n)$, $C(n+1)$ and $D(m,n)$ infinite series and the exceptional $D(2,1;\alpha)$, $G(3)$ and $F(4)$ superalgebras.

$\mathfrak{U}_q(\mathfrak{G})$ is a $\mathbb{Z}_2$-graded algebra which admits a presentation by generators and relations, as in (5-7) with the following alterations. Commutators become graded commutators. The q-adjunction is also graded. There are supplementary relations for some types of Dynkin diagrams[18]. The Hopf superalgebra structure differs from the usual Hopf algebra structure by the supercommutativity of the tensor product:

$$(a \otimes b) = (a \otimes 1)(1 \otimes b) = (-1)^{\deg a \, \deg b}(1 \otimes b)(a \otimes 1) \tag{40}$$

where $\deg x$ is the $\mathbb{Z}_2$ degree of the homogeneous element x. Let ℓ_i be an orthogonal basis of $\mathfrak{H}$, the Cartan subalgebra of $\mathfrak{U}_q(\mathfrak{G})$. We denote by η the restriction of $(.|.)$ to $\mathfrak{H}$: it is non degenerate and of signature (p, p'). The case $A(n,n)$ deserves a special treatment, related to the center of $sl(n|n)$ [14]: for convenience, we will exclude this case in the sequel.

With these modifications, it is easy to check from [14] that we still have the multiplicative formula (15), with now

$$K = q^{\sum_{i,j} \eta^{ij} \ell_i \otimes \ell_j} \quad \text{where} \quad \eta^{ij} = (\eta^{-1})_{ij} \tag{41}$$

$$\hat{R}_\alpha = \exp_{\bar{q}_\alpha}\left((-1)^{\deg\alpha}(q - q^{-1})e_\alpha \otimes f_\alpha\right) \quad \text{with} \quad \bar{q}_\alpha = (-1)^{\deg\alpha}q^{-(\alpha|\alpha)}. \tag{42}$$

The linear equation we consider is identical to (18) provided one uses $B(x) = q^{\sum_{i,j} \eta^{ij}(\ell_i - \mu_i)\ell_j}$. As a result $F(x)$ is still defined by (21), and the algebraic proofs remain identical, apart from the use of the graded tensor products. The analytic results are still satisfied without modification due to the non degeneracy of η.

Example: case of $osp(1|2)$

In the case of $osp(1|2)$, with Cartan generator h, fermionic step operators e and f and Cartan matrix $a^{\mathrm{sym}} = (2)$, we have

$$F_{12}(x) = \sum_{n=0}^{+\infty} e^n \otimes f^n \phi_n(x, h_{(2)}). \tag{43}$$

The recursion relation satisfied by $(-1)^{n(n-1)/2}\phi_n(x,h)$ is the same as the one found for $sl(2)$ provided that we change q^2 into $-q^2$ (except in the factors $(q - q^{-1})^n$ and $q^{-2h_{(2)}}$ which are left unchanged) and x^2 into $-x^2$. Finally,

$$F_{12}(x) = \sum_{n=0}^{+\infty} \frac{(q - q^{-1})^n}{[n]_{-q^2}!} e^n \otimes f^n \frac{(-1)^{n(n+1)/2}}{\prod_{\nu=1}^{n}\left(1 + x^{-2}(-q^2)^\nu q^{-2h_{(2)}}\right)}. \tag{44}$$

and, for x sufficiently large,

$$F(x) = \prod_{k=0}^{+\infty} B_2(x)^k \hat{R}_{12}^{-1} B_2(x)^{-k} = \prod_{k=0}^{+\infty} \exp_{-q^2}\left(-(q - q^{-1})\, x^{-2k}q^{-2k}q^{-2kh_{(2)}}\, e \otimes f\right). \tag{45}$$

5 Conclusion

In our work we have obtained a universal solution of the GNF equations for finite dimensional Lie (super)algebras. There are different paths along which our work can be pursued.

It would be very interesting to generalize our work to the case of affine Lie algebra. A step in this direction has been achieved by C. Frønsdal [11]. This should shed some light on the construction of elliptic solutions of GNF equations.

It is now certainly possible to generalize our work [6] to arbitrary non compact quantum complex group. In particular explicit formulas for $F(x)$ should allow us to understand the proof of the Plancherel formula in these cases.

It has been shown in [2] that in the $sl(2)$ case there exists an element $g(x) \in \mathfrak{U}_q((sl(2))$ such that $F(x) = \Delta(g(x))g_2(x)^{-1}g_1(xq^{\ell_{(2)}})^{-1}$. It is conjectured that in the case of $sl(n)$ there exists $g(x) \in \mathfrak{U}_q((sl(n))$ such that

$$F(x) = \Delta(g(x))Fg_2(x)^{-1}g_1(xq^{\ell_{(2)}})^{-1} \tag{46}$$

where F satisfy the cocycle equation. The exact expression for $F(x)$ should be useful in the understanding of this property.

Finally it would be very interesting to use our framework to derive the eigenfunctions of Ruijsenaars–Schneider system (which are related to Macdonald's polynomials) using purely quantum group techniques.

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